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Observability Considerations for an Inertial Gyrocompass

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An inertial gyrocompassing and platform leveling system, which attempts to infer the attitude errors of an inertial platform by observing the change in measured acceleration as the Earth's gravity vector rotates through the inertial coordinate frame, is investigated with respect to observability. The approach taken is based on the physical effects of a changing gravity vector in an inertial coordinate system. The general equations are posed in terms of generalized gyro drifts, which are transformations of the component drifts. A method is then presented by which steady-state rms error can be determined from the generalized gravity equations.

I. Introduction

An inertial gyrocompassing and platform leveling system is considered, which attempts to infer the attitude errors of an inertial platform by observing the change in measured acceleration as the Earth's gravity vector rotates through the inertial coordinate frame.

In this paper, the basic system is investigated with respect to observability. It is assumed that the system is initially, roughly positioned to a reference and then set inertial never to be repositioned, except to torque out the inferred error angles. In this case, it is shown that the system is not completely observable. In fact, azimuth error, the error in attitude about the local vertical, is limited by acceleration-sensitive gyro drift errors. This should be contrasted with more conventional gyrocompassing techniques in which final azimuth error is limited by the total gyro drift errors as projected onto the east axis. The difference comes about due to the time-varying nature of the "east" axis in an inertial platform, thus allowing some additional separation of terms in situations where sufficient time is available.

The approach taken in this paper is based on the physical effects of a changing gravity vector in an inertial coordinate system. First, the equations are developed for the three time-varying components of gravity in a system which is positioned with the X axis along the Earth's rate vector (pointed at the polar star). These equations clearly show that only twelve distinct combinations of gyro drift and attitude error can be inferred from the acceleration signatures. There are, however, three attitude errors, three bias drifts, and nine acceleration-sensitive drifts. Thus, at least three modes of the system are not observable. In fact, analysis shows that there are exactly four unobservable modes.

It is then shown that the equations for the polar star case are generally valid for a system which is tracking any star. The general equations, however, are posed in terms of generalized gyro drifts. These generalized drifts are transformations of the component drifts, and so are essentially defined as drift coefficients with respect to platform axes rather than component axes.

A method is then presented by which steady-state rms error can be determined from the generalized gravity equations. The method assumes that in steady-state perfect knowledge of the twelve signature coefficients would be attained. Therefore, these twelve coefficients are posed as perfect measurements, and a steady-state covariance matrix is calculated assuming that minimum mean square error estimation is applied.

Results are presented for operation at two latitudes, 15 and 45 deg. For each of these two latitudes, contour plots of final rms azimuth error are presented where initial platform orientation was varied.

II. Derivation of Accelerometer Outputs for North Star

The geometry for determining the nominal components of gravity on the accelerometer axes is shown in Fig. 1. Here, the platform is initially oriented so that the X_0 axis is along the Earth's polar axis, the Y_0 axis is east, and the Z_0 axis forms a right-handed orthogonal set. The initial vertical, north, and east axes are denoted by V_0, N_0, and E_0, respectively. g represents the direction of the gravity vector at some time t after the start of the process and is rotated about the Earth's polar axis by an angle $w_e t$, where w_e is Earth's rate, from the V_0 axis. L is the latitude angle. The nominal components of g on the accelerometer axes are then

$$g_{X_0} = g \sin L$$

$$g_{Y_0} = g \cos L \sin w_e t$$

$$g_{Z_0} = -g \cos L \cos w_e t \qquad (1)$$

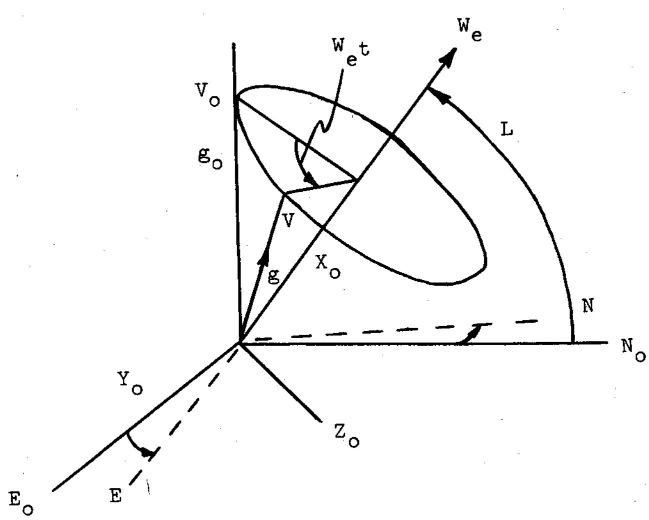

Fig. 1 Geometry for inertial platform aligned with Earth's polar axis.

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The actual components of g along misaligned axes are given by

$$g_X = g_{X_0} + \theta_Y g_{Z_0} - \theta_Z g_{Y_0}$$
$$g_y = g_{y_0} + \theta_Z g_{X_0} - \theta_X g_{Z_0}$$
$$g_Z = g_{Z_0} + \theta_X g_{Y_0} - \theta_Y g_{X_0} \tag{2}$$

Here θ_X, θ_Y, and θ_Z are the misalignment angles about X, Y, and Z axes, respectively, and are assumed to be small. θ_X, θ_Y, and θ_Z are due to initial misalignment angles θ_{X_0}, θ_{Y_0}, θ_{Z_0}, and drift. Then, in general terms, neglecting g-squared and higher-order drift terms.

$$\theta_X = \theta_{X_0} + BD_X t + AD_{XX}\int_0^t g_{X_0}\,dt + AD_{XY}\int_0^t g_{Y_0}\,dt + AD_{XZ}\int_0^t g_{X_0}\,dt$$
$$\theta_Y = \theta_{Y_0} + BD_Y t + AD_{YX}\int_0^t g_{X_0}\,dt + AD_{YY}\int_0^t g_{Y_0}\,dt + AD_{YZ}\int_0^t g_{Z_0}\,dt$$
$$\theta_Z = \theta_{Z_0} + BD_Z t + AD_{ZX}\int_0^t g_{X_0}\,dt + AD_{ZY}\int_0^t g_{Y_0}\,dt + AD_{ZZ}\int_0^t g_{Z_0}\,dt \tag{3}$$

Here, BD_X is bias drift about the X axis, AD_{XX} is drift about the X axis due to acceleration along the X axis, etc. It is assumed that drift and misalignment angles are small enough so that the acceleration-sensitive drift rate can be calculated using nominal components of g on the gyro axes. Using Eqs. (1)

$$\int_0^t g_{X_0}\,dt = gt\,\sin L$$
$$\int_0^t g_{Y_0}\,dt = \frac{g}{w_e}\cos L(1-\cos w_e t)$$
$$\int_0^t g_{Z_0}\,dt = -\frac{g}{w_e}\cos L\,\sin w_e t \tag{4}$$

From Eqs. (2), the difference between the actual and nominal components of g is given by

$$\Delta g_X = g_X - g_{X_0} = \theta_Y g_{Z_0} - \theta_Z g_{Y_0}$$
$$\Delta g_Y = g_Y - g_{Y_0} = \theta_Z g_{X_0} - \theta_X g_{Z_0}$$
$$\Delta g_Z = g_Z - g_{Z_0} = \theta_X g_{Y_0} - \theta_Y g_{X_0} \tag{5}$$

Equations (5) define the measurements used to estimate drifts and initial alignment angles. Substituting Eqs. (4) into Eqs. (3) and using the resulting expression along with Eqs. (1) in Eqs. (5) yields

$$\begin{aligned}\Delta g_X = & -K_1[\theta_{Y_0} + AD_{YY}K_1/w_e]\cos w_e t \\ & -K_1[BD_Y + AD_{YX}K_2]t\cos w_e t \\ & +K_1[AD_{YY}K_1/w_e]\cos^2 w_e t \\ & +K_1[AD_{YZ}K_1/w_e + AD_{ZY}K_1/w_e]\sin w_e t\cos w_e t \quad \text{(cont.)} \\ & -K_1[\theta_{Z_0} + AD_{ZY}K_1/w_e]\sin w_e t \\ & -K_1[BD_Z + AD_{ZX}K_2]t\sin w_e t \\ & +K_1[AD_{ZZ}K_1/w_e]\sin^2 w_e t\end{aligned} \tag{6}$$

$$\begin{aligned}\Delta g_Y = & K_2[\theta_{Z_0} + AD_{ZY}K_1/w_e] + K_2[BD_Z + AD_{ZX}K_2]t \\ & +[K_2(-AD_{ZY}K_1/w_e) + K_1(\theta_{X_0} + AD_{XY}K_1/w_e)]\cos w_e t \\ & +K_2[-AD_{ZZ}K_1/w_e]\sin w_e t \\ & +K_1[BD_X + AD_{XX}K_2]t\cos w_e t \\ & -K_1[AD_{XY}K_1/w_e]\cos^2 w_e t \\ & -K_1[AD_{XZ}K_1/w_e]\sin w_e t\cos w_e t\end{aligned} \tag{7}$$

$$\begin{aligned}\Delta g_Z = & [K_1(\theta_{X_0} + AD_{XY}K_1/w_e) + K_2(AD_{YZ}K_1/w_e)]\sin w_e t \\ & +K_1[BD_X + AD_{XX}K_2]t\sin w_e t \\ & +K_1[-AD_{XY}K_1/w_e]\sin w_e t\cos w_e t \\ & +K_1[-AD_{XZ}K_1/w_e]\sin^2 w_e t - K_2[\theta_{Y_0} + AD_{YY}K_1/w_e] \\ & -K_2[BD_Y + AD_{YX}K_2]t - K_2[AD_{YY}K_1/w_e]\cos w_e t\end{aligned} \tag{8}$$

where $K_1 = (g)\cos L$, and $K_2 = (g)\sin L$.

Consider now that it is possible to observe the gravity profiles for an indefinite period of time, and that some sort of estimation procedure is applied, such as least-squares curve-fitting or Kalman filtering. The quantities which will be observable are the coefficients of each distinct time "signature" [e.g., $\cos(w_e t)$, $t\sin(w_e t)$, etc.]. Note, however, that there are only twelve of the coefficients which are distinct, namely,

$(\theta_{Y_0} + AD_{YY}K_1/w_e)$, $(\theta_{Z_0} + AD_{ZY}K_1/w_e)$, $(BD_Y + AD_{YX}K_2)$, $(BD_Z + AD_{ZX}K_2)$, $(AD_{YZ} + AD_{ZY})$, (AD_{ZZ}), (AD_{YY}), $(-AD_{ZY}K_2/w_e + \theta_{X_0} + AD_{XY}K_1/w_e)$, $(BD_X + AD_{XX}K_2)$, (AD_{XZ}), (AD_{XY}), and $(\theta_{X_0} + AD_{XY}K_1/w_e + AD_{YZ}K_2/w_e)$

III. Equivalence of Dynamics

In this section, it will be shown that a simple transformation of coordinates can be introduced which causes the dynamic equations to be identical in form for any two systems operating at the same latitude. Under the assumption that bias and g-sensitive gyro drifts are the primary errors in the system, this transformation will allow the dynamic model to be stated in terms of "generalized" gyro drifts. In terms of the generalized drifts, then, all systems will appear to be identical, regardless of the star being tracked, provided that they are operating at the same latitude.

If bias and g-sensitive drifts are the only errors driving platform position, then the differential equations for platform attitude error $\overline{\Delta\theta}$ and velocity deviation from nominal $\overline{\Delta V}$ are given by[1]

$$\frac{d}{dt}\overline{\Delta\theta(t)} = \overline{\text{drift}} = \overline{BD} + F_D\overline{g(t)} \tag{9}$$

$$\frac{d}{dt}\overline{\Delta V(t)} = \overline{g(t)} \times \overline{\Delta\theta(t)} \tag{10}$$

where the drift model is defined by the vector of bias drifts $\overline{BD}$ and the matrix of g-sensitive drift coefficients

$$F_D = \begin{bmatrix} ADXX & ADXY & ADXZ \\ ADYX & ADYY & ADYZ \\ ADZX & ADZY & ADZZ \end{bmatrix} \tag{11}$$

Now, consider that two systems are operating at the same latitude, but with different inertial orientations. Thus, the gravity vectors in the two systems are related by a constant direction cosine matrix D;

$$\overline{g_2}(t) = D\overline{g_1}(t) \tag{12}$$

The differential equations for the two systems are

$$\frac{d}{dt}\overline{\Delta\theta_1}(t) = \overline{BD} + F_D\overline{g_1}(t)$$

$$\frac{d}{dt}\overline{\Delta V_1}(t) = \overline{g_1}(t) \times \overline{\Delta\theta_1}(t) \tag{13}$$

$$\frac{d}{dt}\overline{\Delta\theta_2}(t) = \overline{BD} + F_D\overline{g_2}(t)$$

$$\frac{d}{dt}\overline{\Delta V_2}(t) = \overline{g_2}(t) \times \overline{\Delta\theta_2}(t) \tag{14}$$

Now, consider a transformation of variables, $\overline{\Delta\theta_2'}(t) = D^{-1}\overline{\Delta\theta_2}(t)$ and $\overline{\Delta V_2'}(t) = D^{-1}\overline{\Delta V_2}(t)$ $(D^{-1} = D^T)$. Since D is a constant matrix, the differential equations for the second system may be written as

$$\frac{d}{dt}\overline{\Delta\theta_2'}(t) = D^T\frac{d}{dt}\overline{\Delta\theta_2}(t) = D^T\overline{BD} + D^TF_D\overline{g_2}(t)$$

$$\frac{d}{dt}\overline{\Delta V_2'}(t) = D^T\frac{d}{dt}\overline{\Delta V_2}(t) = D^T\overline{g_2}(t) \times D^T\overline{\Delta\theta_2}(t) \tag{15}$$

Since $\overline{g_1}(t) = D^T\overline{g_2}(t)$,

$$\frac{d}{dt}\overline{\Delta\theta_2'}(t) = \overline{D^TBD} + D^TF_DD\overline{g_1}(t) = \overline{BD'} + F_D'g_1(t)$$

$$\frac{d}{dt}\overline{\Delta V_2'}(t) = \overline{g_1}(t) \times \overline{\Delta\theta_2'}(t) \tag{16}$$

where we have defined the generalized drifts

$$\overline{BD'} = D^T\overline{BD} \tag{17}$$

and

$$F_D' = D^TF_DD \tag{18}$$

Obviously, Eqs. (16) are identical in form to Eqs. (13).

IV. Steady-State Accuracy Analysis Technique

In this section, the technique for steady-state error analysis of the platform positioning system is developed. In Sec. II, the acceleration equations were developed for a platform located at an arbitrary latitude, aligned to the North Star. In Sec. III, it was shown that the same dynamics govern a system in any position, but in terms of generalized parameters. Here, the coefficients which were identified as observable in Sec. II are posed in terms of the generalized parameters. These "generalized coefficients" are then taken to be perfect measurements in order to simulate an infinite tracking time. The perfect measurement model is then used to generate an estimation error covariance matrix assuming minimum mean square error estimation. This covariance matrix then represents the limiting accuracies to be achieved while tracking a single star.

In order to pose the estimation problem, however, the system of equations must be modelled in terms of a state vector. To this end, it is desirable to write the generalized g-sensitive drift matrix F_D' as a vector. For the untransformed (North Star) case, this is simply

$$\overline{f_D}^T = [ADXX, ADXY, ADXZ, ADYX, ADYY, ADYZ, ADZX, ADZY, ADZZ] \tag{19}$$

For the generalized case, this can be written as

$$\overline{f_D'} = [DD]\overline{f_D} \tag{20}$$

where

$$DD = \left[\begin{array}{c|c|c} D(1,1)D^T & D(2,1)D^T & D(3,1)D^T \\ \hline D(1,2)D^T & D(2,2)D^T & D(3,2)D^T \\ \hline D(1,3)D^T & D(2,3)D^T & D(3,3)D^T \end{array}\right]$$

and $D(i,j)D^T$ represents the 3×3 matrix D^T multiplied by the (i,j) element of D (a scalar). Now, we can define a reference state vector

$$\overline{X} = \begin{bmatrix} \overline{\Delta\theta_1} \\ \overline{BD} \\ \overline{f_D} \end{bmatrix} \tag{21}$$

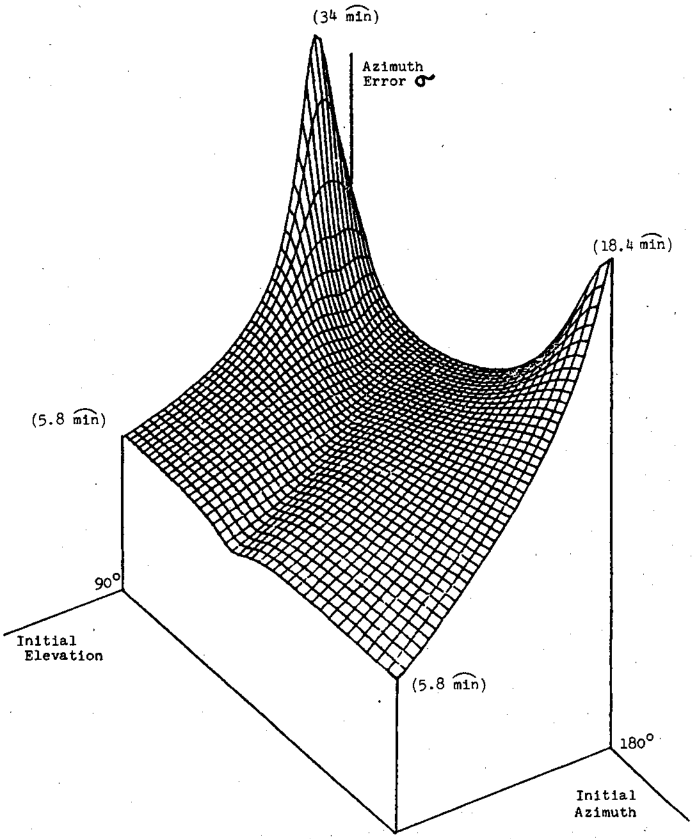

Fig. 2 Azimuth error contour at 15° N latitude.

and a generalized state vector

$$\overline{X'} = \begin{bmatrix} \overline{\Delta\theta'_2} \\ \overline{BD'} \\ \overline{f'_D} \end{bmatrix} \tag{22}$$

Now, the parameters are in a form which facilitates steady-state error analysis. In Sec. II, twelve coefficients were identified as being observable or identifiable. In steady-state these coefficients should be estimated perfectly. Thus, steady-state can be simulated by assuming that perfect measurements of the coefficients are available, and we model a 12×1 measurement vector

$$\overline{Z} = H\overline{X} \tag{23}$$

where $H\,(12 \times 15)$ is given by

$$H = \begin{bmatrix}
0 & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-K_1^2}{W_e} & 0 & 0 & 0 & 0 \\
0 & 0 & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-K_1^2}{W_e} & 0 \\
0 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & 0 & -K_1K_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -K_1 & 0 & 0 & 0 & 0 & 0 & 0 & -K_1K_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_1^2}{W_e} & 0 & \frac{K_1^2}{W_e} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_1^2}{W_e} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_1^2}{W_e} & 0 & 0 & 0 & 0 \\
K_1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_1^2}{W_e} & 0 & 0 & 0 & 0 & 0 & \frac{-K_1K_2}{W_e} & 0 \\
0 & 0 & 0 & K_1 & 0 & 0 & K_1K_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-K_1^2}{W_e} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-K_1^2}{W_e} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
K_1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{K_1^2}{W_e} & 0 & 0 & 0 & \frac{K_1K_2}{W_e} & 0 & 0 & 0
\end{bmatrix} \tag{24}$$

$$K_1 = (g)\cos(L) \qquad K_2 = (g)\sin(L)$$

and $\bar{X}$ is given by Eq. (21). For a platform which is tracking a star other than the North Star, the measurement is the same but in terms of the generalized state vector. Thus,

$$\overline{Z'} = H\overline{X'} \tag{25}$$

The generalized state, however, is simply a linear transformation of $\overline{X}$. Thus,

$$\overline{Z'} = H \begin{bmatrix} D^T & 0 & 0 \\ 0 & D^T & 0 \\ 0 & 0 & DD \end{bmatrix} \overline{X} = H_g\overline{X} \tag{26}$$

Now, for minimum mean square error estimation with perfect measurements, the Kalman filter equations[2] can be used with a null measurement noise covariance matrix. Thus, the final covariance matrix of the state estimation errors P_f is given by

$$P_f = P_0 - P_0H_g(H_gP_0H_g^T)^{-1}H_gP_0 \tag{27}$$

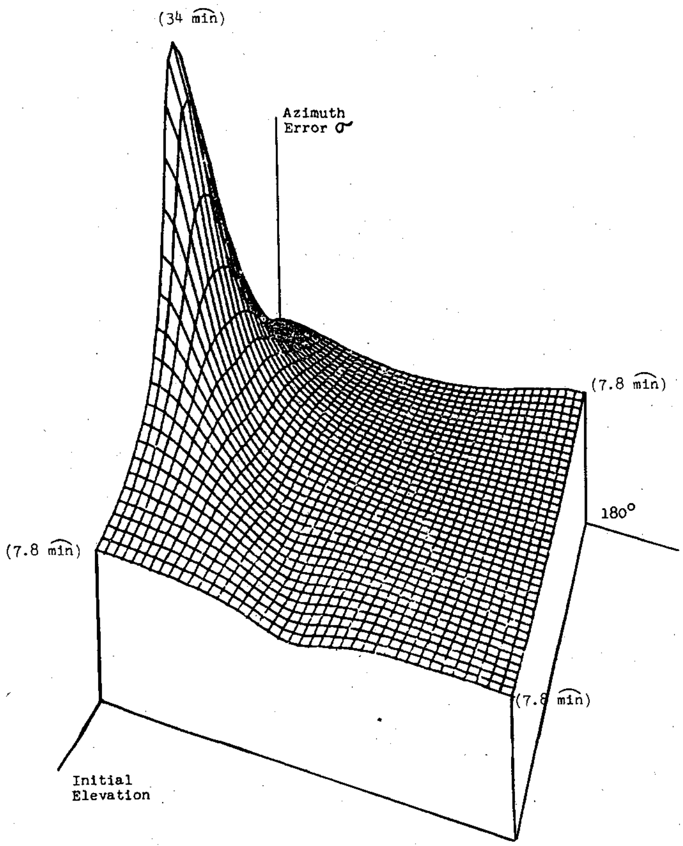

Fig. 3 Azimuth error contour at 45° N latitude.

where P_0 is the initial covariance matrix. Note that, in general, $(H_g P_0 H_g^T)^{-1}$ does not exist. However, the twelve independent measurements can be treated as twelve scalar measurements, and those which contribute no additional information can be ignored.

V. Results

In this section, selected steady-state rms estimation errors will be presented for an inertial platform with bias and g-sensitive gyro drifts and only acceleration or velocity measurements available. The results are presented for two latitudes, 15°N and 45°N. The initial rms errors used were 5.0 deg for the attitudes, 0.15 deg/h for bias drifts, 0.11 deg/h/g for input and output axis g-sensitive drifts, and 0.025 deg/h/g for spin axis g-sensitive drifts. The Y axis is considered to be initially horizontal, so initial orientation is defined by elevation and bearing of the X axis.

Figures 2 and 3 show the contours of final rms azimuth error as a function of initial platform orientation. Note that the peak error always occurs when the X axis is pointed at the North Star. This is attributable to nonuniform gyro drift error statistics. If all g-sensitive drifts had the same initial rms error σ_g, then Figs. 2 and 3 would be flat surfaces reflecting a constant rms azimuth error of (229.2) σ_g, where σ_g is in deg/h/g. The peaks and valleys are due to the nonisotropic nature of the error statistics. Note, however, that rms azimuth error is never reduced to zero.

VI. Practical Considerations

The purpose of this paper has only been to analyze the observability of an inertial alignment system, and thus only those effects which contribute directly to steady-state error have been considered. No consideration has been given to the time required to achieve the theoretical limits. In actual practice, effects such as accelerometer measurement error, base motion of the inertial platform, and randomly varying gyro drifts will dictate the time required to reach steady-state or perhaps even a practical inability to ever reach the limits presented herein.

Even in a nearly perfect environment, achievement of the theoretical accuracy limit is a lengthy process. Because the system dynamics are dependent on a 24-h Earth rotation, several hours may be required in order to allow separation of gyro drift terms. In fact, experience has shown that the short-term response of the preceding system is nearly identical to that of a more conventional gyrocompass. That is, the short term azimuth accuracy is given by $D_\epsilon/(w_e \cos L)$ (see Ref. 3) where D_ϵ represents the total east component of drift errors.

An advantage to be gained by inertial operation, however, is the eventual accuracy to be achieved if time allows. Also, even in short-term operation, an additional accuracy advantage exists in that the system is not torqued to an Earth-fixed reference. Thus, torquing errors, which would normally be indistinguishable from gyro drift errors, do not contribute to azimuth error in this case.

VII. Conclusions

The covariance matrix, which results from the preceding analysis, reveals much about the observability of the system. By inspecting the diagonals of P_f, the final estimation variances for attitudes, bias drifts, and g-sensitive drifts are determined. Inspection of the correlations in P_f reveals which parameters are confounded with each other, or equivalently, the observable modes of the system.

Closer analysis reveals that the rank of H is eleven, or equivalently, that there are four nonobservable modes of the system. The reduction from twelve equations to eleven can be seen from Eq. (24) by row operations on the H matrix, (row 8) + (K_2/K_1) (row 5) − (row 12) = a zero row. Note also that the transformation from H to Hg is unitary. Therefore, rank is preserved and there are exactly four nonobservable modes, regardless of the initial platform orientation.

In fact, transformation of the attitude error piece of the final covariance matrix to a local north, east, and vertical system shows, as expected, that north and east attitudes are always observable and that azimuth is always confounded with some combination of g-sensitive drifts. Therefore, the results of this analysis have been presented in terms of steady-state azimuth error.

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